Investigation on the In-plane Nonlinear Vibration of Cables with Small Sag and Large Sag

by

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Supervisor

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Abstract

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by Kun Wang

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In this thesis, the partial differential equations in time and space about the vibrations of both inclined cables with small sag-to-span ratio and horizontal suspended cable with large sag-to-span ratio are derived. The confusion in deriving the expression of the horizontal component of the cable tensile force in the present cable theory is identified and clarified. Then the deriving procedure and the expression for the horizontal component of the cable tensile force used for hundreds of years are revised. The geometrically nonlinear terms are considered in formulating the equations of motion of cables. Galerkin’s method is applied to make the nonlinear partial differential equations reduced to nonlinear ordinary differential equations (ODEs) or multi-degree-of-freedom nonlinear dynamical systems. Finally, Runge-Kutta method is used to obtain both the linear and nonlinear steady state response of the cables. The convergence of the solutions is investigated as the number of shape functions increases and the frequency of external harmonic excitation changes. The results of
the nonlinear systems and the simplified linear system are compared to show the importance of considering the nonlinearity and analyzing the vibrations of the cables as a multi-degree-of-freedom systems rather than only a single-degree-of-freedom system. For the suspended cable with large sag, the limit of sag-to-span ratio is also investigated and determined in the numerical analysis. The conditions of replacing the differential arc length $ds$ by the horizontal differential length $dx$ are also investigated with some conclusions given.

**Key Words**: cable, sag-to-span ratio, geometrical nonlinearity, steady state response, shape function, multi-degree-of-freedom system.
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