New Numerical Methods and Analysis for Toeplitz Matrices with Financial Applications

by

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Doctor of Philosophy in Mathematics

2011

Faculty of Science and Technology

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Acknowledgments

I would like to express my sincere gratitude to my supervisor, Prof. Xiao-Qing Jin, for leading me to the interesting fields of fast iterative Toeplitz solvers and option pricing problems. His constant encouragement, great support, inspired guidance, and emphasis on international view and elegant English usage (listening, speaking, reading, and writing) are immensely valuable, both in my personal and professional life.

I would also like to show my great appreciation to my co-supervisors, Dr. Hai-Wei Sun, for his enthusiasm, counsel, help, support, and valuable guidance in research, Prof. Deng Ding, for his kindly help, valuable guidance, and enlightening suggestions in financial mathematics fields, and Dr. George Seak-Weng Vong, for his helpful discussions and beneficial suggestions. I am also indebted to Prof. Tao Qian, Prof. Vai-Kuong Sin, Prof. Raymond Che-Man Cheng, and Dr. David Siu-Long Lei for their scholarly talks which broaden my horizons beyond my expertise. My thanks are also due to Prof. Sik-Chung Tam, Dr. Ieng-Tak Leong, and Dr. Anna Kit-Ian Kou in the Mathematics Department. I would also profoundly acknowledge all the staffs and technicians working in the Faculty of Science and Technology for providing considerate assistance and adequate facilities to me.

My special thanks go to my former M.Sc. supervisor, Prof. Wen Li of School of Mathematical Sciences, South China Normal University, for recommending me to University of Macau for my Ph.D. study, and for his patience, inspired guidance, constant encouragement and help throughout the period of my Ph.D. study.

I offer my regards and blessings to my colleagues Ying-Ying Zhang, Spike Tsz-Ho Lee, and Xin Liu for their kind help and support in my Ph.D. journey. Thanks also go to my current and former colleagues in University of Macau. They are Yan-Bo Wang, Pei Dang, Da-Sheng Zhou, Wen Mi, Peng-Tao Li, Shuang Li, Jin-Xun Wang, Qing-Jiang Meng, Shu-Huai Xu, Yue-Lin Liu, Ning-Ying Huang, Zuo-Qiu Weng, Jian Gong,
Chon-Ip Chao, Peter Io-Kei Lok, Mike Kin-Sio Fong, Sio-Chong U, Qi Fu, Zhi-Xiong Li, Tong Zhang, Rui-Hui Xu, Zhu-Lin Liu, Chun-Yang Zhang, Si-Fan Wu, Wei-Xiong Mai, Jing-Ya Zhao, and Rong Shi.

Finally, I would like to express my deepest appreciation to my family for their encouragement, patience, understanding, consideration, and endless love. They are, and will always be, all that matters to me.
Abstract of thesis entitled

New Numerical Methods and
Analysis for Toeplitz Matrices with
Financial Applications

submitted by

Hong-Kui PANG

for the Degree of Doctor of Philosophy in Mathematics
at University of Macau, 2011

Many applications in numerical mathematics, scientific computing, and engineering require solving large-scale matrix problems with Toeplitz structure. Iterative solution techniques based on projection processes onto Krylov subspaces are used to solve those problems. Both the efficiency and robustness of iterative techniques can be improved by using preconditioning. The process of preconditioning is essential to most successful applications of iterative methods. In this thesis, our main goal is to develop preconditioning techniques for solving large-scale Toeplitz systems, Toeplitz matrix exponential, and apply them to option pricing problems in financial engineering.

First, we study the mathematical properties of the optimal preconditioner and the generalized superoptimal preconditioner. Several existing results are extended and new properties are developed.

We also consider using the normalized preconditioned conjugate gradient method with a tri-diagonal preconditioner to solve a nonsymmetric Toeplitz system, which arises from the discretization of a partial integro-differential equation (PIDE) in option pricing problems. By using the definition of family of generating functions introduced in
[116], we prove that the tri-diagonal preconditioner leads to a superlinear convergence rate under certain conditions. Numerical results exemplify our theoretical analysis.

Apart from Toeplitz systems, the Toeplitz matrix exponential (TME) also plays a key role in various application fields. In this thesis, we exploit the Krylov subspace method with the shift-invert preconditioning technique to approximate the TME. By making use of the Toeplitz structure and the famous Toeplitz matrix inversion formula, we reduce the computational cost to $O(n \log n)$ in total compared with the $O(n^3)$ complexity for traditional methods. Moreover, for the nonsymmetric TME, a sufficient condition for the approximation error bound is established, which guarantees that the error bound is independent of the norm of the matrix. Numerical results are given to demonstrate the efficiency of the method.

Finally, we consider approximating a matrix exponential with block Toeplitz matrix, which arises from the integration of a two dimensional PIDE in option pricing under the stochastic volatility jump diffusion model by the exponential time integration scheme. The shift-invert Arnoldi method is employed to fast approximate the matrix exponential. This results in an inner-outer iteration. To reduce the computational cost, we utilize the matrix splitting technique with multigrid method to deal with the shift-invert matrix-vector product in each inner iteration. Numerical results show that the proposed scheme is robust and efficient even compared with the existing high accurate implicit-explicit Euler based extrapolation scheme.
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